Quadratic Voting as Efficient Corporate Governance: Appendices

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APPENDIX A. EX ANTE SHAREHOLDER VALUE AND EX POST DECISIONS

QV is ex post (Kaldor-Hicks) efficient in the sense that, at the moment of decision, it maximizes the prospective wealth of society. The goal of a firm about to make an initial public offering, however, is not to maximize this ex post efficiency but to minimize the cost of raising capital. To compare these two we consider a simple model.

There are two stages. At the first stage, a market for shares exists. At the second stage, a decision is made. There are N individuals with a representative individual i having utility $u_i^A(s_i) - ps_i$ if she owns s_i shares, the price of shares is p, and the decision in the second period is made in favor of the action and utility $u_i^{\neg A}(s_i) - ps_i$ if the decision is made not to implement the action. Thus we assume, for simplicity and because any given corporate holding is likely a small fraction of any individual's lifetime wealth, that utility is quasi-linear in money. We also assume that $u_i^{A\prime}, u_i^{\neg A\prime} > 0 > u_i^{A\prime\prime}, u_i^{\neg A\prime\prime}$ so that individuals value shares but will want to purchase a finite number.

Suppose that, independent of share purchases, the decision is made for A with probability π. Then if individuals evaluate prospects as maximizers of expected utility, individual i's expected utility is $\pi u_i^A(s_i) + (1 - \pi)u_i^{\neg A}(s_i) - ps_i$. If individuals can freely choose the number of shares they buy, then their first-order conditions are $\pi u_i^{A\prime}(s_i) + (1 - \pi)u_i^{\neg A\prime}(s_i) = p$. This must hold for all individuals i, and, because we denominate shares as a fraction of the total shares, they must sum to 1.

Suppose that those running the IPO, whom we will collectively refer to as "the entrepreneur," wish to choose π (or a rule for choosing π) to maximize profits. The implicit function theorem yields that

$$\left[\pi u_i^{A''}(s_i) + (1-\pi)u_i^{\neg A''}(s_i)\right]\frac{ds_i}{d\pi} + u_i^{A'}(s_i) - u_i^{\neg A}(s_i) = \frac{dp}{d\pi}$$

for all i, assuming all individuals buy shares, and $\sum_i \frac{ds_i}{d\pi} = 0$. Thus, solving out,

$$\frac{dp}{d\pi} = \frac{s_i \epsilon_i}{\bar{\epsilon}}\left[u_i^{A'}(s_i) - u_i^{\neg A}(s_i)\right],$$

where the price elasticity of share purchases,

$$\epsilon_i \equiv \frac{p}{s_i[\pi u_i^{A''}(s_i) + (1-\pi)u_i^{\neg A''}(s_i)]} > 0$$

for all i by our assumptions and $\bar{\epsilon}$ is the share-weighted average price elasticity of share purchases across individuals. Thus the initial shareholders want to move toward whichever decision maximizes a weighted average across shareholders of their *marginal utility for additional shares*, in which weights are proportional to the number of shares an individual owns multiplied by their elasticity of their share purchases with respect to price. This is effectively a collective version of the classic Ramsey price-discriminatory rule. The problem is that, once shares have been sold, individuals' interests are determined only by $u_i^A(s_i) - u_i^{\neg A}(s_i)$, the level of their utility, *not by their marginal utility for additional shares*. While these may be aligned under some assumptions, in general they will not be perfectly so. For example, individuals might be happy to see a corporation reduce pollution, even at the cost of some profits per share, but this would not make them more willing to pay for the shares of the corporation. In a subtler but interesting example, a firm taking on a project that is risky but highly profitable might make shareholders better off, but actually less willing to pay for shares at the margin, depending on the nature of risk preferences.

This fundamental problem stops QV, or any ex post efficient mechanism, from maximizing ex ante shareholder value. However it is not just ex post efficient mechanisms that will fail to maximize ex ante shareholder value: an ex post inefficient mechanism like voting is likely to do even worse as there is no reason at all that what is preferred by the majority of shares

should align with efficiency. For example if shareholders believe, as is possible under majority voting, that one or a few individuals will buy enough shares to vote for the inefficient expropriation of minority shareholders, the equilibrium price of votes will fall to zero as the marginal value of shares is zero in the state when such an expropriation takes place. More broadly, Professors Philippe Jehiel and Benny Moldovanu show that the only information that a mechanism can elicit from agents is that information that directly affects their payoffs from the action that the mechanism will determine.[1] That is, *any* ex post mechanism followed by free trading in shares (so that the mechanism cannot force agents to rearrange their share holdings) can never elicit complete information about which decision maximizes ex ante shareholder value. We believe that maximizing ex post efficiency is likely to line up with ex ante shareholder value more reliably under QV than under any other mechanism in a broader range of cases and that ex post efficiency is desirable in itself. However, there will no doubt be cases when ex post inefficient actions might nonetheless promote ex ante shareholder value.

Another argument we discuss in the text is that linking votes to shares might improve the share price. To show why we are skeptical of this claim, imagine that π is now a function of the number of shares each individual owns. From the perspective of individual i, only the dependence of π on her own shares is decision relevant, and thus we abuse notation slightly by writing $\pi(s_i)$. Individual i's decision-relevant utility is then $\pi(s_i)u_i^A(s_i) + [1 - \pi(s_i)]u_i^{\neg A}(s_i) - ps_i$. Now individual i's first-order condition is

$$\pi u_i^{A'}(s_i) + (1 - \pi)u_i^{\neg A'}(s_i) + \pi'(s_i)[u_i^A(s_i) - u_i^{\neg A}(s_i)] = p.$$
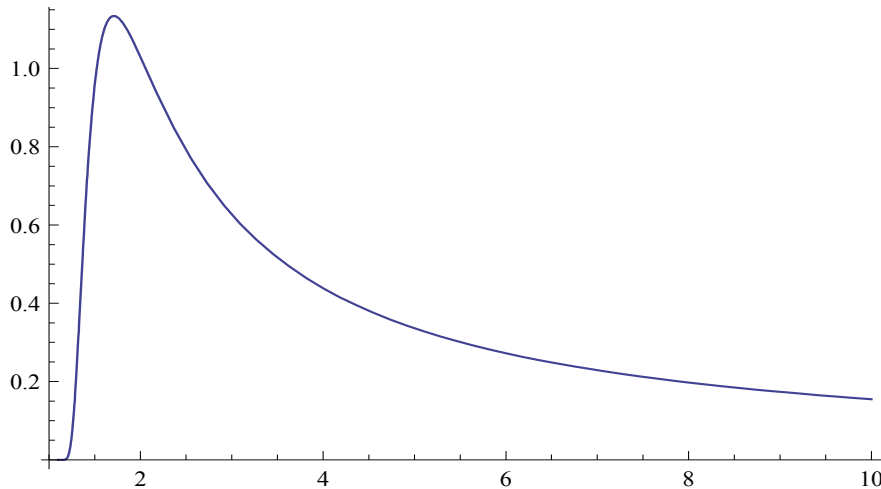
Thus selling influence over the decision can raise the price of shares to the extent that the most marginal influence is granted to those who gain the most (ex post) by influencing the decision. This seems to indicate that those who already have a lot of shares (and thus a lot at stake) should be offered the most influence in exchange for purchasing an additional share. However, three things are worth noting.

First, there is no sense in which this component of the price is maximized by the one-share–one-vote rule, or by any simple

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[1]   See Philippe Jehiel and Benny Moldovanu, *Efficient Design with Interdependent Valuations*, 69 Econometrica 1237, 1240–41 (2001).

linking of votes to shares. Thus the current system is hardly geared to raising maximal revenue off of votes.

Second, a series of recent theorems by most notably Professors George Mailath and Andrew Postlewaite and most generally Professors Nabil Al-Najjar and Rann Smorodinsky imply that as N grows large, so long as each s_i becomes small (no one owns a large part of the firm), π' must become small for all individuals i, so much so that the amount of revenue that can be raised in total is very small compared to the total value of the firm.[2] Thus the revenue raised by linking voting to shares is likely to be very small compared to the value created by getting the decision "right." While again, the "right" decision from the perspective of ex ante shareholder value need not be the ex post efficient decision, this seems (as argued above) more likely than any other ex post rule to maximize ex ante shareholder value.

Finally, note that the contribution to the price of the voting rights enters linearly into the equation. Thus, especially given the logic above, it seems sensible to consider what voting system (separate from shares) would raise maximal revenue (even if this is very small, perhaps even irrelevant, by the previous discussion) and thus maximally raise the value of shares (based on distributing this revenue back to shareholders to raise their willingness to pay for shares).

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[2]   See George J. Mailath and Andrew Postlewaite, *Asymmetric Information Bargaining Problems with Many Agents*, 57 Rev Econ Stud 351, 354–59 (1990); Nabil I. Al-Najjar and Rann Smorodinsky, *Pivotal Players and the Characterization of Influence*, 92 J Econ Theory 318, 323–32 (2000).

Weyl has considered this question in work that is not public-ly available but is available on request. He calibrates a model of preferences that depend on both income and an idiosyncratic component, in which income follows the log-normal distribution with parameters approximating the income distribution in the United States and an idiosyncratic component to preferences that makes individuals on average willing to pay roughly $\frac{1}{1000}$ of their annual income to make the decision go their way. He con-siders vote-buying rules in which the price of votes is propor-tional to v^x, where v is the number of votes purchased. $x = 2$ is QV, $x = 1$ is linear vote buying (as occurs with shares), and $x = \infty$ is democracy. The figure above shows revenues as a function of x. The peak does not occur at 2, but slightly below. However, 2 is very close to the peak and far above either the values of 1 or ∞. Thus QV seems likely to raise at the very least as much rev-enue from the allocation of votes as does share-linked voting, likely much more than that and close to the maximal revenue one could achieve by selling off votes to maximize revenue at least in plausible circumstances.

APPENDIX B. QV'S BIAS AGAINST SMALL-GROUP OPPORTUNISM

The theoretical results Weyl establishes about QV depend on the assumption that every individual is "small" in the sense that he or she has independently and identically distributed preferences and there are a large number of individuals.[3] In cor-porate governance, on the other hand, there may be a single in-dividual or a small group of individuals who have strong prefer-ences potentially opposite to those of the large mass of the popu-lation. For example, an outside raider or do-gooder, or the CEO, may seek to take an action that is against the interest of most (by numbers of people, not necessarily number of shares) share-holders. Such opportunism, anticipated ex ante, will lower the willingness of individuals to pay for shares and thus, by the logic of Appendix A, lower the profits. In this Appendix we show that such an attempt by a single individual, which we label "oppor-tunism," is never advantageous. The individual would always do better to make payments to the other shareholders such that the transaction is in the others' interests rather than to attempt to

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[3]   E. Glen Weyl, *Quadratic Vote Buying* (University of Chicago Working Paper, Apr 1, 2013), online at http://ssrn.com/abstract=2003531 (visited Oct 18, 2013).

vote in the opportunistic scheme themselves. In this sense, QV is opportunism-proof.

We consider only the case when there is a single opportunistic individual. The same argument could easily be extended so long as the size of the opportunistic group is small (of constant size even as the size of other voters/shareholders grows). In particular, suppose that an individual proposes an opportunistic action A. There is a large number of shareholders N who have values drawn independently and identically from some distribution with negative mean μ and variance σ^2 for action A occurring; they thus typically are harmed by the opportunistic action.[4] Let M_i be the ith moment of the distribution of small individual valuations; we assume the first four moments of the distribution exist. The opportunist has a (for simplicity commonly known) utility $U > 0$ from A being undertaken. For large N, A is weakly efficient if and only if $U \geq -N\mu$ and inefficient otherwise. To analyze this case we draw heavily on the techniques of Weyl without introducing them pedagogically here.[5] Those interested in following the argument more closely should first read at least Sections 1 and 2 of that paper.

Following the logic of Subsection 2.2 of Weyl, we approximate the votes purchased by any small individual with utility u by $v(u) = a(N)u + b(N)u^2$. The mean of votes purchased by all small individuals together is then $m \equiv Na(N)\mu + Nb(N)(\mu^2 + \sigma^2)$ and their variance

$$s^2 \equiv N(a^2(N)[\mu^2 + \sigma^2] + a(N)b(N)[M_3 - \mu(\mu^2 + \sigma^2)] + b^2(N)[M_4 - (\mu^2 + \sigma^2)^2]\,).$$

The distribution of the sum of all N small individuals' votes is approximately normal with mean m and variance s^2, while the distribution of the sum of any $N-1$ individuals votes is approximately normal with mean $\frac{N-1}{N}m$ and variance $\frac{N-1}{N}s^2$.

Let g be the distribution of the sum of all N small individual votes. The opportunist will buy votes satisfying $v_o^* = \frac{g(-v_o^*)}{2}U \approx \dfrac{e^{-\frac{(v_o^*+m)^2}{2s^2}}}{2s\sqrt{2\pi}}U$. The distribution of votes by everyone else facing any

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[4]   Note that the iid assumption is not important here as long as all individuals are small. For example, if different individuals had different shares and these scaled up their values, but values were otherwise iid, as long as no individual had a large number of shares the analysis would proceed along precisely the same lines.

[5]   Weyl, *Quadratic Vote Buying* at *6–15 (cited in note 3).

small individual is approximately normal and has mean $\frac{N-1}{N}m +$ v_0^* and variance $\frac{N-1}{N}s^2$ because U is common knowledge and thus v_0^* deterministic (or essentially 0 if the opportunist plays a mixed strategy). Thus, by the logic of the proof of Lemma 2 in the Appendix of Weyl,[6]

$$\frac{b(N)}{a(N)} = -\frac{\frac{N}{N-1}v_0^* + m}{2s^2}$$

and

$$a(N) = \frac{p(N)\sqrt{N}e^{-\frac{\left(\frac{N}{N-1}v_0^*+m\right)^2}{2\frac{N}{N-1}s^2}}}{2s\sqrt{2\pi(N-1)}},$$

where $p(N)$ is the probability that the opportunist tries to buy the election if she is playing a mixed strategy (and 1 if she always tries).[7]

There are three cases to consider: that $\frac{v_0^*}{m}$ heads towards 0 for large N; that as N grows large, v_0^* grows in absolute value relative to m without bound; and that $\frac{v_0^*}{m}$ converges to a constant. Let us consider these cases in order. First, if $\frac{v_0^*}{m}$ shrinks as N grows large, clearly the opportunist wins with vanishing probability (given that s shrinks relative to m by the law of large numbers) as N grows large, so our case is already proven.

In the second case,

$$a(N) \to \frac{p(N)e^{-\frac{N(v_0^*)^2}{2(N-1)s^2}}}{2s\sqrt{2\pi}},$$

$$\frac{b(N)}{a(N)} \to -\frac{v_0^*}{2s^2},$$

and

$$v_o^* \to \frac{e^{-\frac{(v_0^*)^2}{2s^2}}}{2s\sqrt{2\pi}}U.$$

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6   Id at *48–49.
7   Id at *13–14.

Thus

$$m \to \frac{Np(N)e^{-\frac{N(v_0^*)^2}{2(N-1)s^2}}}{2s\sqrt{2\pi}}\left[\mu - \frac{v_0^*}{2s^2}(\mu^2 + \sigma^2)\right]$$

and

$$\frac{m}{v_0^*} \to \frac{Np(N)e^{-\frac{(v_0^*)^2}{2(N-1)s^2}}}{U}\left[\mu - \frac{v_0^*}{2s^2}(\mu^2 + \sigma^2)\right].$$

Clearly in order for our hypothesis that $\frac{m}{v_0^*}$ vanishes for large N to be maintained either $p(N)$ must vanish (in which case clearly there is no limiting chance of the opportunist's victory) or $\frac{(v_0^*)^2}{2(N-1)s^2}$ must explode in N. But

$$s^2 \geq Na^2(N)(\mu^2 + \sigma^2) \to \frac{p^2(N)Ne^{-\frac{N(v_0^*)^2}{(N-1)s^2}}}{8s^2\pi} \Rightarrow s = \Omega\left(e^{-\frac{N(v_0^*)^2}{4(N-1)s^2}}\right).$$

Thus $v_0^* = O\left(e^{-\frac{(N-2)(v_0^*)^2}{4(N-1)s^2}}\right)$. But this implies that the number of votes purchased by the opportunist dies exponentially with N. This clearly cannot be an equilibrium as any other shareholder would then pay to outvote the opportunist on her own. It can be shown that there is no other equilibrium supported by such behavior for reasons tightly analogous to those in the constant $\frac{v_0^*}{m}$ case to which we now turn.

If $\frac{v_0^*}{m}$ approaches a constant, then again there are three subcases, though only one that is interesting. If $\frac{v_0^*}{m} \to \gamma < 1$ then the probability of victory conditional on buying votes by the opportunist is less that $\frac{1}{2}$, which is ruled out by the fact that, as shown in Subsection 2.3 of Weyl,[8] such a strategy will never be optimal for the opportunist. If $\gamma > 1$ then a logic very similar to the one above in the case where $\frac{v_0^*}{m}$ explodes shows this cannot be an equilibrium. In the case in which $\frac{v_0^*}{m} \to 1$, we can index events by the finite value to which $\frac{\frac{N}{N-1}v_0^* + m}{s}$ converges to; if the value is in-

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8   Id at *13–15.

finite then again the analysis resembles the case when $\frac{v_0^*}{m}$ explodes and is thus omitted. Call the finite limiting value $\hat{\gamma}$. Then

$$a(N) \to \frac{p(N)\sqrt{N}e^{-\frac{\hat{\gamma}^2(N-1)}{2N}}}{2s\sqrt{2\pi(N-1)}}$$

and

$$s^2 \geq Na^2(N)(\mu^2 + \sigma^2) \to \frac{p^2(N)e^{-\frac{\hat{\gamma}^2(N-1)}{2N}}}{8s^2\pi} \Rightarrow s = \Omega\left(\sqrt[4]{N}\sqrt{p(N)}\right).$$

As above, if $p(N) \to 0$ as for large N, then we are done. Suppose this is not the case. Then because $\frac{b(N)}{a(N)} \to 0$ as N grows large in this case by the arguments in the proof of Lemma 2 in Weyl, it is also the case that $s = O\left(\sqrt[4]{N}\right)$. Thus $a(N) = \Omega\left(\frac{1}{\sqrt[4]{N}}\right)$. Thus

$$m \leq Na(N)\mu = \Omega\left(\mu N^{\frac{3}{4}}\right).$$

But $\frac{v_0^*}{m} \to 1$ implies that $v_0^* = \Omega\left(\mu N^{\frac{3}{4}}\right)$. Because the cost of votes is quadratic, this means the opportunist would have to be expending an amount on votes that is $\Omega\left(\mu^2 N^{\frac{3}{2}}\right)$. Clearly this is only worthwhile if $U = \Omega\left(\mu^2 N^{\frac{3}{2}}\right)$ as the best the opportunist can do is switch the decision's outcome for sure at the cost of this many votes. But given that the total utility of the small individuals is only of size $N\mu$, as $N\mu$ grows large this requires an unboundedly large amount of surplus to be generated by the opportunist's desired course of action. For example, suppose there are 10,000,000 shareholders and each has a disutility of $50 on average from the action taking place, as seems plausible for a major corporate decision. They thus in aggregate have a disutility on average of $500 million of the action taking place. Then the opportunist would have to have utility from the decision on the order of $80 trillion for an equilibrium where the opportunist wins with non-vanishing probability to prevail. Clearly such an outcome is not a serious possibility for large decisions.

Thus, the only equilibrium is qualitatively similar to the one discussed in Subsection 2.3 of Weyl. [9] The opportunist, with probability $p(N)$ that vanishes with N, buys a substantial proba-

9   Weyl, *Quadratic Vote Buying* at *13–15 (cited in note 3).

bility of winning the decision. But now this probability is generated by a mixed strategy of the opportunist rather than the tail probabilities of the distribution of small individuals. Note that this means that whenever the opportunist buys a substantial number of votes, she is indifferent between doing so and buying no votes at all. Thus her expenditures must equal the fraction of *U* equal to the probability with which she, conditional on buying these shares, wins the vote. Given that all of this revenue is distributed to individuals *other* than the opportunist, all individuals other than the opportunist will be weakly better off as a result of this purchase if and only if it is efficient for the opportunist to win.

So clearly the opportunist has nothing to gain by trying to hijack the process in equilibrium when the transaction is large. A much easier strategy is for the opportunist to adjust the offer to promise the other, small shareholders a utility, which is positive on average by paying them a fraction of *U* contingent on approving the plan. If $U > -N\mu$ this is always feasible while still leaving positive surplus to the opportunist. For example, the opportunist could promise to pay $\frac{U - N\mu}{2}$ to the small shareholders if the transaction is approved, leaving her with surplus utility $\frac{U - N\mu}{2}$. Because QV is efficient for large *N*, this proposal will be approved (even if the opportunist buys no votes at all!) with probability 1 if there are many small shareholders. This will yield the opportunist $\frac{U - N\mu}{2}$, much better than the 0 she would earn trying to hijack the process. Thus, at least with a large number of shareholders, an efficient opportunist maximizing utility will always choose to pay off small shareholders, thus making them better off, rather than trying to defeat them under QV.[10]

Intuitively the reason for this, reflected in the logic of the constant limiting $\frac{v_0^*}{m}$ case above, is that the quadratic nature of

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[10]   Other strategies exist as well. The opportunist could make such a deal with a subset of shareholders. However, this is more expensive for the opportunist than is the other strategy because of the underdog effect (a demonstration is available on request). Another strategy would be to offer side payments to nonshareholders, or larger side payments to small than to large shareholders. This could be effective by expanding the pool of those interested in voting in the opportunist's interests. However, such strategies could easily be ruled out as tantamount to fraudulent collusion/side payments in the charter, just as other forms of fraud and collusion would have to face legal sanction. Individuals should be restricted from making side payments that are not divided evenly among shareholders per share.

the cost of votes makes it prohibitively expensive to stand alone against the world, even if the rest of the world has less total utility than you do. It is cheaper simply to induce others to agree with you and allow them to, at much lower costs, vote in the proposal that benefits you. Given that they are diffuse and cannot effectively bargain, it should be particularly easy for the opportunist to offer them the minimum amount necessary to ensure his victory so she can keep the maximum amount of surplus. Small shareholders are still better off as a result of this, however. Thus opportunism is not a concern under QV and the appraisal remedy is therefore unnecessary.