Appendix to *The Efficiency of Bargaining under Divided Entitlements*

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Here we derive the results for the example considered in the text. We first assume that Alpha's benefit x is distributed uniformly on $[0, 100]$ and that Smith's harm y is distributed on $[0, 100]$ according to the distribution function $F(y) = (y/100)^a$ with $a > 0$, which corresponds to the density function $f(y) = F'(y) = ay^{a-1}/100^a$. Then, we have:

- Average harm from pollution: $\int_0^{100} yf(y)dy = \frac{ay^{a+1}}{(a+1)100^a}\Big|_{y=0}^{100} = \frac{100a}{a+1}$.

- Expected efficient surplus given x: $\int_0^x (x-y)f(y)dy = xF(x) - \int_0^x yf(y)dy = \frac{x^{a+1}}{100^a} - \frac{ax^{a+1}}{(a+1)100^a} = \frac{1}{a+1}\frac{x^{a+1}}{100^a}$.

- Overall efficient expected surplus: $\int_0^{100} \frac{1}{a+1}\frac{x^{a+1}}{100^a}\left(\frac{1}{100}\right)dx = \frac{100}{(a+1)(a+2)}$.

- Probability of pollution in efficient outcome: $\int_0^{100} F(x)\left(\frac{1}{100}\right)dx = \frac{1}{a+1}$.

A. Property Rules

- The required subsidy when Smith has the right to clean water is $\frac{100}{(a+1)(a+2)}$.

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- The required subsidy when Alpha has the right to pollute is $\frac{100}{(a+1)(a+2)} - (E[x] - E[y]) = \frac{100}{(a+1)(a+2)} - 50 + \frac{100a}{a+1} = \frac{50a}{a+2}$.

Note that both subsidies are positive. Smith is the better owner if and only if $a \geq 1$.

B. Liability Rules

1. Alpha is the chooser.

Suppose that Alpha can choose to pollute and pay a damage D to Smith. If the damage payment is $D \geq \frac{100a}{a+1}$, then type 0 is Alpha's adverse opt-out type, giving Smith a payoff of 0. Alpha's average liability-rule payoff (for any level of Smith's harm) is $(100 - D)^2/200$ since Alpha chooses to pollute with probability $(100 - D)/100$, and its average benefit net of the damage payment when it does pollute is $(100 - D)/2$. The required subsidy is therefore

$$\frac{100}{(a+1)(a+2)} - (100 - D)^2/200,$$

which is increasing in D.

If, instead, the damage payment is $D \leq \frac{100a}{a+1}$, then Alpha's adverse opt-out type has a benefit of 100 (and so always pollutes under the liability rule), giving Smith a payoff of $D - \frac{100a}{a+1}$. Alpha's average liability-rule payoff is again $(100 - D)^2/200$. So the required subsidy is

$$\frac{100}{(a+1)(a+2)} - \left(D - \frac{100a}{a+1}\right) - (100 - D)^2\left(\frac{1}{200}\right),$$

which is decreasing in D.

So the subsidy-minimizing damage payment is $D = \frac{100a}{a+1}$ (in other words, damages equal to the expected harm), which yields subsidy

$$\frac{100}{(a+1)(a+2)} - \left(100 - \frac{100a}{a+1}\right)^2 \left(\frac{1}{200}\right).$$

2. Smith is the chooser.

Now suppose that Smith has the right to insist on clean water in return for a payment D. Smith's expected payoff from this liability rule is

$$-D\big(1-F(D)\big)-\int_0^D yf(y)dy = -D\left(1-\frac{D^a}{100^a}\right)-\frac{a}{a+1}\frac{D^{a+1}}{100^a}=\frac{D}{a+1}\left(\frac{D^a}{100^a}-(a+1)\right),$$

since he chooses to pay D when $y \geq D$, which happens with probability $[1-F(D)]$ and otherwise incurs harm y.

If $D \leq 50$, then Smith's adverse opt-out type is 100, giving Alpha a payoff of D. The subsidy is then

$$\frac{100}{(a+1)(a+2)}-D-\frac{D}{a+1}\left(\frac{D^a}{100^a}-(a+1)\right)=\frac{100}{(a+1)(a+2)}-\left(\frac{D^{a+1}}{a+1}\right)\left(\frac{1}{100^a}\right),$$

which is decreasing in D.

If $D \geq 50$, then Smith's adverse opt-out type is 0, giving Alpha an expected payoff of 50. The required subsidy is then

$$\frac{100}{(a+1)(a+2)}-50-\frac{D}{a+1}\left(\frac{D^a}{100^a}-(a+1)\right)$$

whose derivative with respect to D is $1-\frac{D^a}{100^a}>0$.

So the subsidy-minimizing damage payment again has damages equal to the expected harm: $D = 50$. This yields expected subsidy

$$\frac{100}{(a+1)(a+2)}-\frac{100}{(a+1)2^{a+1}}.$$

Thus, Alpha should be the chooser if and only if

$$\left(100-\frac{100a}{a+1}\right)^2\left(\frac{1}{200}\right)-\frac{100}{(a+1)2^{a+1}}=\frac{50}{(a+1)^2 2^a}(2^a-a-1)\geq 0,$$

which holds if and only if $a \geq 1$. Thus, for $a > 1$, it is better for Alpha to be the chooser, and for $a < 1$, it is better for Smith to be the chooser.

C. Two Victims

Smith's adverse opt-out type is D_S, Jones's adverse opt-out type is D_J. Jones's harm is distributed on $[0, 100]$ according to probability density $g(z) = (bz^{b-1}/100^b)$, where $b > 0$. Alpha's benefit is now drawn from the uniform distribution on $[0, 200]$.

- *Smith + Jones coalition value*: In parallel to our discussion in the two-agent case, Alpha's adverse opt-out type is 200 if $D_S + D_J \leq \frac{100a}{a+1}+\frac{100b}{b+1}$ and is 0 if $D_S + D_J \geq \frac{100a}{a+1}+\frac{100b}{b+1}$. The coalitional value of Smith + Jones is then $\min\left\{\frac{100a}{a+1}+\frac{100b}{b+1}, D_S + D_J\right\}$.

- *Smith + Alpha coalition value when Jones has harm* D_J: Given values of x and y, this coalition pollutes if $x - D_J \geq y$. So the coalition's expected payoff given x is $\left(\frac{1}{a+1}\right)\left(\frac{1}{100^a}\right)(x - D_J)^{a+1}$ if $x - D_J \in [0, 100]$, $[(x - D_J) - \frac{100a}{a+1}]$ if $x - D_J \geq 100$, and 0 otherwise. So the coalition's expected payoff as a function of D_J is

$$V_{-Jones}(D_J) = \left(\frac{1}{200}\right)\left[\int_{D_J}^{100+D_J}(x - D_J)^{a+1}\left(\frac{1}{a+1}\right)\left(\frac{1}{100^a}\right)dx + \int_{100+D_J}^{200}[(x - D_J) - \frac{100a}{a+1}]dx\right] = \left(\frac{1}{200}\right)\left[\int_0^{100} t^{a+1}\left(\frac{1}{a+1}\right)\left(\frac{1}{100^a}\right)dt + \int_{100-\frac{100a}{a+1}}^{200-D_J-\frac{100a}{a+1}} t\,dt\right] \quad (4)$$

- *Jones + Alpha coalition value when Smith has harm* D_S: Parallel to the previous derivation, this coalition's expected payoff as a function of D_S is:

$$V_{-Smith}(D_S) = \left(\frac{1}{200}\right)\left[\int_{D_S}^{100+D_S}(x - D_S)^{b+1}\left(\frac{1}{b+1}\right)\left(\frac{1}{100^b}\right)dx + \int_{100+D_S}^{200}\left[(x - D_S) - \frac{100a}{a+1}\right]dx\right]$$

Taking the derivative of $V_{-Jones}(D_J)$ we see that:

$$V'_{-Jones}(D_J) = -\left[1 - \frac{D_J}{200} - \frac{a}{2(a+1)}\right] = -\frac{1}{2}\left(1 + \frac{1}{a+1} - \frac{D_J}{100}\right)$$

Note, first, that $D_J \leq 100$ implies that $V'_{-Jones}(D_J) < 0$, while $D_J \geq 0$ implies that $V'_{-Jones}(D_J) > -1$. The same is true for $V'_{-Smith}(D_S)$. When $D_S + D_J \geq \frac{100a}{a+1} + \frac{100b}{b+1}$, the first coalition value is unaffected by the damage payments, while the second and third are decreasing in them. So, among such pairs (D_S, D_J) it is best to set $D_S + D_J = \frac{100a}{a+1} + \frac{100b}{b+1}$. On the other hand, when $D_S + D_J \leq \frac{100a}{a+1} + \frac{100b}{b+1}$ the derivative of the first coalition value with respect to either damage payment is 1, while the derivatives of the second and third coalition values are each greater than −1. Hence, among all such damages, it is best to set $D_S + D_J = \frac{100a}{a+1} + \frac{100b}{b+1}$. Thus, the total damages should equal the average total harm, exactly as when bargaining is not possible.

However, while only the total damage matters when bargaining is impossible, when imperfect bargaining is possible the required subsidy can be affected by the division of damages among the victims. We wish to choose these damage payments to solve

Maximize $V_{-Jones}(D_J) + V_{-Smith}(D_S)$ subject to $D_S + D_J = \frac{100a}{a+1} + \frac{100b}{b+1}$.

Examining the expression for $V'_{-Jones}(D_J)$ we see that this derivative is increasing in D_J, so $V_{-Jones}(D_J)$ is a convex function. The same is true of $V_{-Smith}(D_S)$. As a result, all of the damages should be given to one of the two victims. Which one should it be? In the example in the main text, the optimal total damages are $\frac{100a}{a+1} + \frac{100b}{b+1} = 100$, and we write using (4)

$$[\,V_{-Jones}(100) + V_{-Smith}(0)] - [\,V_{-Jones}(0) + V_{-Smith}(100)] = \int_{100-\frac{100b}{b+1}}^{200-\frac{100b}{b+1}} t\,dt - \int_{100-\frac{100a}{a+1}}^{200-\frac{100a}{a+1}} t\,dt.$$

In the text we have assumed $b < a$, in which case the expression is negative and so it is optimal to set $D_J = 100$ and $D_S = 0$.